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Axial and transverse Stokes flow past slender axisymmetric bodies

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This paper deals with Stokes flow due to a stationary axially symmetric slender body in a uniform stream, which may be either parallel or perpendicular to the axis of the body. The effect of the body is represented by distributions of singularities along a segment of its axis of symmetry. Systems of linear integral equations for these distributions are obtained, and the first few terms of uniformly valid (in the Stokes region) asymptotic expansions in the slenderness ratio are discussed. The leading terms yield the expected result that the drag on the body in a transverse stream is double that in an axial stream. The second approximation to the ratio of these two drags is also independent of the body shape.

1. Introduction

It has been conjectured for some years that if the Reynolds number is very small, a slender body of revolution falls twice as fast axially as it does transversely. The phenomenon has been illustrated by Taylor (1967), who has also (Taylor 1969) given a theoretical study of the problem. Otherwise, apart from the classical case of the spheroid and some important pioneering work of Burgers (1938) on straight cylinders, work on slow flow past slender bodies seems to have been confined to axial motion. Reliable experimental data also seem to be lacking.

We consider the Stokes flow past a slender axisymmetric body whose equation is

$$r \equiv (x^2+y^2)^{\frac{1}{2}} = \epsilon R(z) \quad (-a \leqslant z \leqslant a), \tag{1.1}$$

where

$$\max_{-a\leqslant z\leqslant a} R(z) = a. \tag{1.2}$$

It is assumed that the body has rounded nose and tail; more specifically, we assume that

$$R^2(z) \sim A(a-z) \quad (A \neq 0) \tag{1.3}$$

as $z \to a$, together with a similar condition as $z \to -a$. Certain smoothness requirements are imposed on $R(z)$ in § 3.

The flow is represented by distributions of singularities along the axis of the body, as was done by Tuck (1964) in the case of axial flow. It is shown that, to lowest order at least, this leads to an approximate solution of our boundary-value problem (for the Stokes equations) that is valid uniformly over the flow field,†

† We are not concerned in this paper with the far (Oseen) field, but only with the Stokes solution.

including the regions near the ends of the body. Handelsman & Keller (1967) have shown that in the analogous potential problem the asymptotic solution can be carried out to all orders, and there seems to be no reason to doubt that this is the case for Stokes flow also.

An analysis of this kind was used by Taylor (1969), who formulated approximate integral equations for the stokeslet distributions in the two cases of axial and transverse flow and found that the operator in the transverse case was half that in the axial case. Since the drag on the body is the sum over the stokeslet distribution of the appropriate elementary drags, this shows that for a given speed of descent the drag experienced by the body in the transverse case is double that in the axial case. However, Taylor's work contains an error in sign in one term in one of the operators which invalidates this reasoning. (See (3.14) and (6.11) below, and the remark after (6.12).) In the present work we develop a more elaborate analysis which shows that the dominant parts of the two operators (they dominate by a factor $\ln \epsilon$) do in fact differ by a factor of two; and in § 6 we find that Taylor's conclusion that the ratio of the drags equals two is valid, though only to logarithmic order.

As in slender-body potential theory, it is necessary to restrict the singularities to a proper subset of the interval $-a \leqslant z \leqslant a$. Failure to observe this precaution leads to singularities in the velocity field at the ends of the body. The technique used by Handelsman & Keller (1967) is applied in § 4 to find the endpoints of the distribution in the axial case, but it is perhaps more direct to use explicitly, following Moran (1963), the condition of uniform validity of the solution near the ends of the body; this is outlined in § 5.

2. Axial flow: the exact problem

We consider the Stokes flow past the body (1.1) when a velocity $(0, 0, -U)$ at infinity is prescribed. The effect of the body on this uniform flow is represented by a distribution of stokeslets of strength $f(z)$, $-a < \alpha \leqslant z \leqslant \beta < a$, and a distribution of irrotational sources of strength $g(z)$, $-a < \gamma \leqslant z \leqslant \delta < a$, the axis of the stokeslets being along the axis Oz of the body.

The velocity components due to a single such stokeslet of unit strength located at the origin are

$$u = \frac{zr}{(z^2+r^2)^{\frac{3}{2}}}, \quad w = \frac{2z^2+r^2}{(z^2+r^2)^{\frac{3}{2}}} \tag{2.1}$$

in the r and z directions respectively. We introduce a stream function ψ defined by

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = \frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{2.2}$$

so that the stream function for a stokeslet of unit strength is

$$\psi = \frac{r^2}{(r^2+z^2)^{\frac{1}{2}}}. \tag{2.3}$$

The corresponding expressions for an irrotational source are

$$u = \frac{r}{(z^2+r^2)^{\frac{3}{2}}}, \quad w = \frac{z}{(z^2+r^2)^{\frac{3}{2}}}, \quad \psi = 1 - \frac{z}{(z^2+r^2)^{\frac{1}{2}}}. \tag{2.4}$$

The functions f and g are determined by the boundary conditions on the body; these can be taken to be that the perturbation velocity and stream function satisfy

$$w(r = \epsilon R(z), z) = U, \quad \psi(r = \epsilon R(z), z) = \tfrac{1}{2}U\epsilon^2 R^2, \tag{2.5}$$

on $-a \leqslant z \leqslant a$. When w and ψ in these equations are written in terms of f and g we obtain a pair of simultaneous integral equations to be satisfied on $-a \leqslant z \leqslant a$, namely

$$U = \int_\alpha^\beta \frac{2(z-\hat{z})^2+\epsilon^2R^2}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}} f(\hat{z})\, d\hat{z} + \int_\gamma^\delta \frac{(z-\hat{z})\, g(\hat{z})\, d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}}, \tag{2.6}$$

$$\tfrac{1}{2}U\epsilon^2R^2 = \epsilon^2R^2 \int_\alpha^\beta \frac{f(\hat{z})\, d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{1}{2}}} - \int_\gamma^\delta \frac{(z-\hat{z})\, g(\hat{z})\, d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{1}{2}}}. \tag{2.7}$$

These are the equations given by Tuck (1970). In deriving (2.7) we have used the fact that the total source strength must vanish:

$$\int_\gamma^\delta g(\hat{z})\, d\hat{z} = 0. \tag{2.8}$$

The work that follows in §§ 3 to 5 will be concerned with the solution of these equations.

3. Axial flow: the equations to first order

In this section we seek the lowest-order solutions of (2.6) and (2.7) in the form

$$f(z,\epsilon) = f_0(z,\epsilon) + O(\epsilon^2), \quad g(z,\epsilon) = g_0(z,\epsilon) + O(\epsilon^2). \tag{3.1}$$

To do this we require approximations to the integrals occurring in (2.6) and (2.7). The form of (2.7) suggests that $g_0 = 0$; this is confirmed in § 4 and we defer consideration of the integrals in (2.7) until then. To lowest order, that is when terms of algebraic order in ϵ are ignored, we are concerned only with the stokeslet distribution, and the equation for this reduces to

$$U = \int_\alpha^\beta \frac{2(z-\hat{z})^2+\epsilon^2R^2}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}} f(\hat{z})\, d\hat{z}, \tag{3.2}$$

which we decompose as

$$U = \int_0^{z-\alpha} \frac{2s^2+\epsilon^2R^2}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}} f(z-s)\, ds + \int_0^{\beta-z} \frac{2s^2+\epsilon^2R^2}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}} f(z+s)\, ds. \tag{3.3}$$

We require an expansion for $\epsilon \to 0$ of each integral in (3.3). With the stretching transformation $s = \epsilon\sigma$, and the notation

$$h(s) = f(z-s), \quad k(\sigma) = \frac{\sigma(2\sigma^2+R^2)}{(\sigma^2+R^2)^{\frac{3}{2}}}, \tag{3.4}$$

the first integrand in (3.3) may be written (with the z-dependence suppressed)

$$\Phi(z,s,\epsilon) = (1/s)\, h(s)\, k(\sigma). \tag{3.5}$$

(The singular factor $1/s$ is isolated for later convenience.) For $\epsilon \to 0$ with s fixed and positive, we expand $k(s/\epsilon)$ up to ϵ^n and obtain, on reintroducing σ, the 'outer' expansion

$$E_n \Phi \equiv (1/s)\, h(s)\, E_n\, k(s/\epsilon)$$
$$\equiv \frac{1}{s}\, h(s) \left\{ k_0 + \frac{k_1}{\sigma} + \ldots + \frac{k_n}{\sigma^n} \right\}, \tag{3.6}$$

where E_n is the outer-expansion operator defined in general by Fraenkel (1969*a*), which in our case effects the operation described above. The coefficients k_m in (3.6) are found by expanding $k(\sigma)$ in (3.4) for $\sigma \to \infty$.

Similarly, for $\epsilon \to 0$ with σ fixed and finite, we expand $h(\epsilon\sigma)$ up to ϵ^n and obtain, on reintroducing s, the 'inner' expansion

$$H_{n-1} \Phi \equiv (1/s)\, k(\sigma)\, H_n\, h(\epsilon\sigma)$$
$$\equiv (1/s)\, k(\sigma) \{ h_0 + h_1 s + \ldots + h_n s^n \}, \tag{3.7}$$

where H_n is the inner-expansion operator, and the coefficients h_m are found from (3.4).

We now form the 'composite' expansion

$$C_{nn} \Phi = (1/s) \{ h E_n k + k H_n h - (H_n h)(E_n k) \}; \tag{3.8}$$

its error then satisfies

$$s(\Phi - C_{nn}\Phi) = (h - H_n h)(k - E_n k)$$
$$= O(s^{n+1} \sigma^{-n-1} |f^{(n+1)}|_{\max}) = O(\epsilon^{n+1} |f^{(n+1)}|_{\max}), \tag{3.9}$$

and this holds uniformly over s and σ; moreover as $\sigma \to 0$ we have no singular behaviour:

$$\Phi - C_{nn}\Phi = O(s^n \sigma^{-n} |f^{(n+1)}|_{\max}) = O(\epsilon^n |f^{(n+1)}|_{\max}). \tag{3.10}$$

Thus, there exists A independent of s, σ, ϵ (but not of z) such that

$$|\Phi - C_{nn}\Phi| \leqslant A\, |f^{(n+1)}|_{\max}\, \epsilon^n/(1+\sigma). \tag{3.11}$$

Note that f depends on ϵ. Our construction of $C_{nn}\Phi$ is equivalent to the method of Handelsman & Keller (1967), but is a little more direct because of Fraenkel's (1969*b*) observation that the formalism of inner and outer expansions is appropriate to integrands having a product structure like that of Φ.

The details of our procedure for approximating the integrals is given in appendix A, where it is confirmed that the remainder term resulting from the error (3.11) is small compared to the terms resulting from $C_{nn}\Phi$. This work can be carried to arbitrary order provided f is sufficiently smooth; from (3.11) it is seen that expansion to nth order requires $f \in C^{n+1}[\alpha, \beta]$,† and this imposes smoothness requirements on $R(z)$. The expressions obtained below in (3.17) to (3.19) are consistent with this requirement if $\ln\{(a^2 - z^2)^{\frac{1}{2}}/R(z)\} \in C^2[-a, a]$. It seems likely that the expansion will be valid to all orders if

$$\ln\{(a^2 - z^2)^{\frac{1}{2}}/R(z)\} \in C^\infty[-a, a].$$

† Here $C^{n+1}[\alpha, \beta]$ denotes as usual the space of functions with continuous $(n+1)$th order derivatives on $[\alpha, \beta]$. No confusion should arise with C_{nn} above.

Working to order $n = 0$ only we have

$$E_0 k = 2, \quad H_0 h = f(z),$$

$$C_{00}\,\Phi = \frac{2f(z-s)}{s} + f(z)\frac{2s^2+\epsilon^2R^2}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}} - \frac{2f(z)}{s}. \tag{3.12}$$

Similar treatment is applied to the second integral in (3.3). This equation then becomes

$$U = 2\int_0^{z-\alpha}\frac{f(z-s)-f(z)}{s}\,ds + 2\int_0^{\beta-z}\frac{f(z+s)-f(z)}{s}\,ds$$
$$+2f(z)\left\{\int_0^{z-\alpha} + \int_0^{\beta-z}\frac{2s^2+\epsilon^2R^2}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}}\,ds\right\} + O(\epsilon^2 f \ln\epsilon). \tag{3.13}$$

On evaluating the last two integrals and approximating them by neglecting contributions which are $O(\epsilon^2 \ln\epsilon)$ uniformly, and rearranging the first two integrals, we obtain the lowest-order form (with terms of algebraic order in ϵ ignored) of our equation for the stokeslet distribution $f(z)$:

$$U = \left\{4\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{\epsilon R} - 2\right\} f_0(z) + 2\int_\alpha^\beta \frac{f_0(\hat{z})-f_0(z)}{|\hat{z}-z|}\,d\hat{z}. \tag{3.14}$$

This is essentially the equation given by Tuck (1964). It has been assumed following Handelsman & Keller (1967), that

$$\alpha = -a + O(\epsilon^2), \quad \beta = a + O(\epsilon^2). \tag{3.15}$$

The solution of the integral equation (3.14) is a matter of some difficulty. One possible procedure is suggested by Tuck (1964); this is to express $f_0(z)$ as an infinite series of Legendre polynomials, (3.14) giving an infinite system of linear algebraic equations for the coefficients. We will content ourselves here with finding the first four terms of an expansion in powers of $(\ln\epsilon)^{-1}$; we write

$$f_0(z,\epsilon) = (\ln\epsilon)^{-1} f_{01}(z) + (\ln\epsilon)^{-2} f_{02}(z) + \ldots, \tag{3.16}$$

and substitution in (3.14) yields

$$f_{01}(z) = -\tfrac{1}{4}U, \tag{3.17}$$

$$f_{02}(z) = -\tfrac{1}{4}U(\ln[2(a^2-z^2)^{\frac{1}{2}}/R] - \tfrac{1}{2}), \tag{3.18}$$

$$f_{03}(z) = -\tfrac{1}{4}U(\ln[2(a^2-z^2)^{\frac{1}{2}}/R] - \tfrac{1}{2})^2$$
$$-\tfrac{1}{8}U\int_{-a}^{a}\frac{\ln[(a^2-\hat{z}^2)^{\frac{1}{2}}/R(\hat{z})] - \ln[(a^2-z^2)^{\frac{1}{2}}/R(z)]}{|\hat{z}-z|}\,d\hat{z}. \tag{3.19}$$

Higher terms can be found from the recurrence relation

$$f_{0,m+1}(z) = \left(\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{R} - \tfrac{1}{2}\right) f_{0m}(z) + \frac{1}{2}\int_{-a}^{a}\frac{f_{0m}(\hat{z})-f_{0m}(z)}{|\hat{z}-z|}\,d\hat{z}. \tag{3.20}$$

The dominant term is thus independent of the shape $R(z)$ of the body; this is because, to this order, we are effectively reducing the axial velocity to zero along the axis $-a \leqslant z \leqslant a$ of the body. Higher terms depend on $R(z)$.

In the special case of the spheroidal body

$$R(z) = (a^2 - z^2)^{\frac{1}{2}}, \tag{3.21}$$

the term $4\ln[2(a^2-z^2)^{\frac{1}{2}}/\epsilon R]$ in (3.14) is a constant, so we can solve exactly for f_0:

$$f_0(z,\epsilon) = U/(4\ln(2/\epsilon)-2). \tag{3.22}$$

The total force on the (general) body is a drag

$$D = 8\pi\mu\int_\alpha^\beta f(z)\,dz \tag{3.23}$$

(where μ is the coefficient of viscosity), the irrotational sources making no contribution. The above results thus give

$$D = 4\pi\mu Ua\left\{-(\ln\epsilon)^{-1} - \tfrac{1}{2}(\ln\epsilon)^{-2}\left(\int_{-a}^{a}\ln[2(a^2-z^2)^{\frac{1}{2}}/R]\,dz - 1\right) + O[(\ln\epsilon)^{-3}]\right\} \tag{3.24}$$

in the general case, and

$$D = 4\pi\mu Ua/(\ln(2/\epsilon)-\tfrac{1}{2}) + O(\epsilon^2\ln\epsilon) \tag{3.25}$$

for the spheroid. The dominant term in (3.24) was given by Tuck (1964). The spheroid result agrees with that of Lamb (1932, p. 605), obtained by solving the Stokes equations by separation of variables.

4. Axial flow: the equations to second order

In this section we approximate (2.6) and (2.7) up to and including terms of order ϵ^2. For the expansion of the first integral in (2.6) this entails the use of (A 9) and (A 11) with $N = 1$. This gives

$$\int_\alpha^\beta \frac{2(z-\hat{z})^2+\epsilon^2R^2}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}} f(\hat{z})\,d\hat{z} = \mathscr{A}_0 f + \epsilon^2\mathscr{A}_{2a}f + \epsilon^2R^2\mathscr{A}_{2b}f + O(\epsilon^4 f\ln\epsilon), \tag{4.1}$$

where the operators $\mathscr{A}_0$, $\mathscr{A}_{2a}$, $\mathscr{A}_{2b}$ are defined by

$$\mathscr{A}_0 f \equiv \left\{4\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{\epsilon R} - 2\right\} f(z) + 2\int_\alpha^\beta \frac{f(\hat{z})-f(z)}{|\hat{z}-z|}\,d\hat{z}, \tag{4.2}$$

$$\mathscr{A}_{2a} f \equiv -2\left(\frac{\alpha_2}{a+z} + \frac{\beta_2}{a-z}\right) f(z), \tag{4.3}$$

$$\mathscr{A}_{2b} f \equiv \left\{\frac{1}{(a+z)^2} + \frac{1}{(a-z)^2}\right\} f(z) - 2\int_\alpha^\beta \frac{f(\hat{z})-f(z)-(\hat{z}-z)f'(z)-\frac{1}{2}(\hat{z}-z)^2f''(z)}{|\hat{z}-z|^3}\,d\hat{z}$$
$$+\frac{4z}{a^2-z^2}f'(z) - \left\{2\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{\epsilon R} - \tfrac{3}{2}\right\} f''(z). \tag{4.4}$$

These expressions are obtained by evaluating, correct to order ϵ^2, those integrals in (A 9) and (A 11) which do not contain f. $\mathscr{A}_0$ is just the lowest-order form of the operator which was found in § 3. $\mathscr{A}_{2a}$ is the correction to this arising from the fact that, in passing from the last two integrals in (3.13) to (3.14), or $\mathscr{A}_0 f$, we took β and α to be $\pm a$; here we have expanded α and β in powers of ϵ as

$$\alpha = -a+\epsilon^2\alpha_2+\ldots, \quad \beta = a-\epsilon^2\beta_2-\ldots. \tag{4.5}$$

$\mathscr{A}_{2b}f$ results from the expansion of the integrand. So far we have expanded only the integral operator acting on $f(z,\epsilon)$, not the function f itself.

For the second integral in (2.6) and both integrals in (2.7) it is only necessary to operate to lowest order; the work thus proceeds along the lines of § 3. The details are given in appendix B, where it is shown that

$$\int_\gamma^\delta \frac{(z-\hat{z})\,g(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}} = \frac{2z}{a^2-z^2}g(z)-2\left\{\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{\epsilon R}-1\right\}g'(z)$$

$$-\int_\gamma^\delta \frac{g(\hat{z})-g(z)-(\hat{z}-z)\,g'(z)}{(\hat{z}-z)^2}\operatorname{sgn}(\hat{z}-z)\,d\hat{z}+O(\epsilon^2 g\ln\epsilon), \tag{4.6}$$

$$\int_\alpha^\beta \frac{f(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{1}{2}}} = 2f(z)\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{\epsilon R}+\int_\alpha^\beta\frac{f(\hat{z})-f(z)}{|\hat{z}-z|}\,d\hat{z}+O(\epsilon^2 f\ln\epsilon), \tag{4.7}$$

$$\int_\gamma^\delta \frac{(z-\hat{z})\,g(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{1}{2}}} = -\int_\gamma^\delta g(\hat{z})\operatorname{sgn}(\hat{z}-z)\,d\hat{z}+O(\epsilon^2 g\ln\epsilon). \tag{4.8}$$

With these results we can now write (2.6) and (2.7) to second order; (2.6) becomes

$$U = \mathscr{A}_0 f+\epsilon^2\mathscr{A}_{2a}f+\epsilon^2R^2\mathscr{A}_{2b}f+\mathscr{B}_0 g, \tag{4.9}$$

where

$$\mathscr{B}_0 g \equiv \frac{2z}{a^2-z^2}g(z)-2\left\{\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{\epsilon R}-1\right\}g'(z)$$

$$-\int_\gamma^\delta \frac{g(\hat{z})-g(z)-(\hat{z}-z)\,g'(z)}{(\hat{z}-z)^2}\operatorname{sgn}(\hat{z}-z)\,d\hat{z}. \tag{4.10}$$

Equation (2.7) is most conveniently written in differential form; this then yields

$$\frac{d}{dz}(\tfrac{1}{2}U\epsilon^2R^2) = \frac{d}{dz}\left[\epsilon^2R^2\left\{2f(z)\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{\epsilon R}+\int_\alpha^\beta\frac{f(\hat{z})-f(z)}{|\hat{z}-z|}\,d\hat{z}\right\}\right]-2g(z)$$

$$= \frac{d}{dz}[\epsilon^2R^2(\tfrac{1}{2}\mathscr{A}_0 f+f)]-2g(z), \tag{4.11}$$

and since $\mathscr{A}_0 f_0 = U$ we obtain

$$g(z) = \frac{d}{dz}(\tfrac{1}{2}\epsilon^2R^2 f)+O(\epsilon^4\ln\epsilon). \tag{4.12}$$

On substituting this expression into (4.9) an equation for f alone is obtained. If we write

$$f = f_0+\epsilon^2 f_2+\ldots, \tag{4.13}$$

where $f_0(z,\epsilon)$ is assumed known from § 3, (4.9) becomes an equation for $f_2(z,\epsilon)$, namely

$$-\mathscr{A}_0 f_2 = \mathscr{A}_{2a}f_0+R^2\mathscr{A}_{2b}f_0+\mathscr{B}_0 g_2, \tag{4.14}$$

where

$$g_2(z,\epsilon) = d(\tfrac{1}{2}R^2 f_0)/dz. \tag{4.15}$$

The method of Handelsman & Keller (1967) may now be applied to evaluate α_2 and β_2. These are determined by the condition that no singularity occurs at $z = \pm a$ in the right-hand member of (4.14). This gives

$$-2\beta_2 f_0(z) + \frac{R^2}{a-z} f_0(z) + g_2(z) = O(a-z) \tag{4.16}$$

as $z \to a$, together with a similar condition for $z \to -a$. If we assume analytic behaviour for R^2, f_0 and g_2 near $z = a$, so that

$$R^2(z) = b_1(a-z) + b_2(a-z)^2 + \dots, \tag{4.17}$$

$$f_0(z) = c_0^{(0)} + c_1^{(0)}(a-z) + \dots, \tag{4.18}$$

$$g_2(z) = d_0^{(2)} + d_1^{(2)}(a-z) + \dots, \tag{4.19}$$

(4.16) in conjunction with (4.15) gives

$$\beta_2 = \tfrac{1}{4} b_1. \tag{4.20}$$

In the special case of the spheroid (3.19) we obtain $\beta_2 = \frac{1}{2}$, which shows that to this order the limits of the stokeslet distribution are the foci of the body. It is anticipated that this will hold to all orders, since in this case singularities may be expected to occur along the singular line of the transformation to spheroidal co-ordinates (see Tuck 1964). In this special case we can solve (4.14) exactly for f_2; (4.15) gives

$$g_2(z) = -z f_0 = -Uz/(4 \ln(2/\epsilon) - 2), \tag{4.21}$$

and (4.14) reduces to

$$-(4 \ln(2/\epsilon) - 2) f_2 = (2 \ln(2/\epsilon) - 2) f_0, \tag{4.22}$$

assuming f_2 independent of z, which gives

$$f_2 = -\frac{2 \ln(2/\epsilon) - 2}{(4 \ln(2/\epsilon) - 2)^2} U, \tag{4.23}$$

confirming the above assumption. The drag to this order is

$$D = \frac{8\pi\mu U a}{2 \ln(2/\epsilon) - 1} \left\{ 1 - \frac{4 \ln(2/\epsilon) - 3}{4 \ln(2/\epsilon) - 2} \epsilon^2 \right\}, \tag{4.24}$$

which again agrees with the results of Lamb (1932).

5. Axial flow: uniformity near the ends

In this section it is shown explicitly that the stokeslet distribution as specified in § 3, but confined to the limits determined in § 4, in conjunction with the source distribution given by (4.12), provides a first approximation valid throughout the (Stokes) flow field, provided that the limits of the source distribution are the same as those of the stokeslet distribution:

$$\gamma_2 = \alpha_2, \quad \delta_2 = \beta_2. \tag{5.1}$$

Away from the ends of the body the radial velocity due to the stokeslets is small compared to the axial velocity, and the stokeslet distribution of § 3 alone provides a first approximation; the sources are required only as a higher-order correction. Near the ends, however, the stokeslets and sources make comparable contributions, and of course the gaps between the ends of the distributions and the ends of the body are crucial.

We have to verify that (2.6) and (2.7) are satisfied near the ends, to a suitable degree of approximation. Assuming

$$R^2(z) = b_1(a-z)+b_2(a-z)^2+\dots, \tag{5.2}$$

$$f(z) = c_0+c_1(a-z)+\dots, \tag{5.3}$$

$$g(z) = d_0+d_1(a-z)+\dots \tag{5.4}$$

near $z = a$, (2.6) and (2.7) become

$$\begin{aligned} U = {} & \int_\alpha^\beta \frac{2f(\hat{z})\,d\hat{z}}{a-\hat{z}}+\int_\gamma^\delta \frac{g(\hat{z})\,d\hat{z}}{(a-\hat{z})^2} \\ & +(a-z)\left\{\int_\alpha^\beta\left[\frac{2f(\hat{z})}{(a-\hat{z})^2}-\frac{2\epsilon^2 b_1 f(\hat{z})}{(a-\hat{z})^3}\right]d\hat{z}+\int_\gamma^\delta\left[\frac{2g(\hat{z})}{(a-\hat{z})^3}-\frac{3}{2}\frac{\epsilon^2 b_1 g(\hat{z})}{(a-\hat{z})^4}\right]d\hat{z}\right\}+O[(a-z)^2], \end{aligned} \tag{5.5}$$

$$\begin{aligned} \tfrac{1}{2}U = {} & \int_\alpha^\beta \frac{f(\hat{z})\,d\hat{z}}{a-\hat{z}}+\frac{1}{2}\int_\gamma^\delta \frac{g(\hat{z})\,d\hat{z}}{(a-\hat{z})^2} \\ & +(a-z)\left\{\int_\alpha^\beta\left[\frac{f(\hat{z})}{(a-\hat{z})^2}-\frac{1}{2}\frac{\epsilon^2 b_1 f(\hat{z})}{(a-\hat{z})^3}\right]d\hat{z}+\int_\gamma^\delta\left[\frac{g(\hat{z})}{(a-\hat{z})^3}-\frac{3}{8}\frac{\epsilon^2 b_1 g(\hat{z})}{(a-\hat{z})^4}\right]d\hat{z}\right\}+O[(a-z)^2]. \end{aligned} \tag{5.6}$$

In each of these integrals the dominant contribution comes from the neighbourhood of $\hat{z} = \beta$ or $\hat{z} = \delta$; to lowest order in ϵ we obtain, on equating coefficients of $a-z$,

$$U = -2c_0\{\ln(a-\beta)+O(1)\}+d_0/(a-\delta), \tag{5.7}$$

$$0 = 2c_0\left\{\frac{1}{a-\beta}-\frac{1}{2}\frac{\epsilon^2 b_1}{(a-\beta)^2}\right\}+d_0\left\{\frac{1}{(a-\delta)^2}-\frac{1}{2}\frac{\epsilon^2 b_1}{(a-\delta)^3}\right\}, \tag{5.8}$$

$$0 = c_0\left\{\frac{1}{a-\beta}-\frac{1}{4}\frac{\epsilon^2 b_1}{(a-\beta)^2}\right\}+d_0\left\{\frac{1}{2}\frac{1}{(a-\delta)^2}-\frac{1}{8}\frac{\epsilon^2 b_1}{(a-\delta)^3}\right\}. \tag{5.9}$$

On using (4.12) to relate d_0 to c_0, we can conclude that (5.8) and (5.9) are satisfied provided that

$$\beta_2 = \delta_2 = \tfrac{1}{4}b_1. \tag{5.10}$$

Equation (5.7) yields no explicit information, though since $c_0 = O[(\ln\epsilon)^{-1}]$ it does confirm that we were correct in assuming $a-\beta = o(1)$.

The work of this section provides an alternative method for determining β_2 and δ_2. Moran (1963) in his work on the corresponding potential-flow problem finds the limits of his source distribution in this way; he points out that, by taking higher powers of $a-z$ in his analogue of (5.5) and (5.6), higher approximations to β may be obtained. In our work, however, the situation is complicated by the presence of two distributions of singularities, which means that (5.5) and (5.6) determine β and δ only when something is already known of f and g; in particular, as we have seen, the determination of β_2 and δ_2 requires knowledge of the second-order $g_2(z)$. To find β_4 and higher terms would require correspondingly higher-order knowledge of f and g. The work of this section does, however, determine δ_2, which is not known from § 4.

6. Transverse flow

We now consider the corresponding transverse-flow problem for the slender body $r = \epsilon R(z)$, when the incident stream U is in the x direction rather than along the axis Oz of the body. We denote by θ the azimuthal angle measured from the x axis.

It is found that three distributions of singularities are required along the axis of the body: a distribution $l(z)$ of stokeslets with axes pointing in the x direction, a distribution $m(z)$ of irrotational source doublets, also with axes in the x direction, and a distribution $n(z)$ of 'Stokes doublets', that is, pairs of infinitesimally separated equal and opposite stokeslets, with stokeslet axes in the z direction and displacement axes in the x direction. This last type of singularity does not appear to have been used previously. These three distributions of singularities seem to be the correct means for describing the flow, as they lead to a system of three linear integral equations, which are *independent of* θ, for the functions $l(z)$, $m(z)$ and $n(z)$. These equations are analogous to (2.6) and (2.7) in the axial-flow case.

With foregoing choice of stokeslet and doublet axes, the velocity components for a stokeslet at the origin are given in cylindrical co-ordinates by

$$\left.\begin{aligned} q_r &= \frac{z^2+2r^2}{(z^2+r^2)^{\frac{3}{2}}}\cos\theta, \quad q_\theta = -\frac{1}{(z^2+r^2)^{\frac{1}{2}}}\sin\theta, \\ q_z &= \frac{zr}{(z^2+r^2)^{\frac{3}{2}}}\cos\theta, \end{aligned}\right\} \tag{6.1}$$

for an irrotational source doublet by

$$\left.\begin{aligned} q_r &= \left\{\frac{1}{(z^2+r^2)^{\frac{3}{2}}} - \frac{3r^2}{(z^2+r^2)^{\frac{5}{2}}}\right\}\cos\theta, \\ q_\theta &= -\frac{1}{(z^2+r^2)^{\frac{3}{2}}}\sin\theta, \quad q_z = -\frac{3zr}{(z^2+r^2)^{\frac{5}{2}}}\cos\theta, \end{aligned}\right\} \tag{6.2}$$

and for a Stokes doublet by

$$\left.\begin{aligned} q_r &= \left\{\frac{3zr^2}{(z^2+r^2)^{\frac{5}{2}}} - \frac{z}{(z^2+r^2)^{\frac{3}{2}}}\right\}\cos\theta, \\ q_\theta &= \frac{z}{(z^2+r^2)^{\frac{3}{2}}}\sin\theta, \quad q_z = \left\{\frac{4r}{(z^2+r^2)^{\frac{3}{2}}} - \frac{3r^3}{(z^2+r^2)^{\frac{5}{2}}}\right\}\cos\theta. \end{aligned}\right\} \tag{6.3}$$

The incident stream has components $(-U\cos\theta, U\sin\theta, 0)$. The boundary condition of zero velocity on the body thus gives us the three integral equations for l, m, n:

$$\begin{aligned} U = \int_\kappa^\lambda \frac{(z-\hat{z})^2+2\epsilon^2R^2}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}} l(\hat{z})\,d\hat{z} &+ \int_\mu^\nu \left[\frac{1}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}} - \frac{3\epsilon^2R^2}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{5}{2}}}\right] m(\hat{z})\,d\hat{z} \\ &+ \int_\rho^\sigma \left[\frac{3\epsilon^2R^2}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{5}{2}}} - \frac{1}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}}\right](z-\hat{z})\,n(\hat{z})\,d\hat{z}, \end{aligned} \tag{6.4}$$

$$U = \int_\kappa^\lambda \frac{l(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{1}{2}}} + \int_\mu^\nu \frac{m(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}} - \int_\rho^\sigma \frac{(z-\hat{z})\,n(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}}, \tag{6.5}$$

$$0 = \epsilon R \int_\kappa^\lambda \frac{(z-\hat{z})\,l(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}} - 3\epsilon R \int_\mu^\nu \frac{(z-\hat{z})\,m(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{5}{2}}} + \int_\rho^\sigma \left[\frac{4\epsilon R}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}} - \frac{3\epsilon^3R^3}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{5}{2}}}\right] n(\hat{z})\,d\hat{z}, \tag{6.6}$$

where κ, λ, μ, ν, ρ, σ are the endpoints of the distributions.

The integrals occurring in (6.4) to (6.6) can be expanded to lowest order (neglecting terms algebraically smaller than the first) along the lines indicated in § 3 and appendix B. As a result we obtain for the lowest-order form of (6.4) to (6.6)

$$U = \left\{2\ln\frac{(a^2-z^2)^{\frac{1}{2}}}{\epsilon R} + 2\right\} l(z) + \int_\kappa^\lambda \frac{l(\hat{z})-l(z)}{|\hat{z}-z|}\,d\hat{z} - \frac{2}{\epsilon^2R^2}\,m(z) + O(\epsilon^2 l\ln\epsilon,\ m\ln\epsilon,\ n\ln\epsilon), \tag{6.7}$$

$$U = 2\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{\epsilon R}\,l(z) + \int_\kappa^\lambda \frac{l(\hat{z})-l(z)}{|\hat{z}-z|}\,d\hat{z} + \frac{2}{\epsilon^2R^2}\,m(z) + O(\epsilon^2 l\ln\epsilon,\ m\ln\epsilon,\ n\ln\epsilon), \tag{6.8}$$

$$0 = \epsilon R\left[\frac{2z}{a^2-z^2}\,l(z) - 2\left\{\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{\epsilon R} - 1\right\} l'(z) - \int_\kappa^\lambda \frac{l(\hat{z})-l(z)-(\hat{z}-z)\,l'(z)}{(\hat{z}-z)^2}\,\operatorname{sgn}(\hat{z}-z)\,d\hat{z}\right] - \frac{2}{\epsilon R}\,m'(z) + \frac{4}{\epsilon R}\,n(z) + O(\epsilon^3 l\ln\epsilon,\ \epsilon m\ln\epsilon,\ \epsilon n\ln\epsilon). \tag{6.9}$$

Subtracting (6.8) from (6.7), we obtain

$$m(z) = \tfrac{1}{2}\epsilon^2R^2 l_0(z) + O(\epsilon^4 l_0\ln\epsilon), \tag{6.10}$$

and (6.7) then becomes

$$U = \left\{2\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{\epsilon R} + 1\right\} l_0(z) + \int_\kappa^\lambda \frac{l_0(\hat{z})-l_0(z)}{|\hat{z}-z|}\,d\hat{z}, \tag{6.11}$$

where we have written

$$l(z,\epsilon) = l_0(z,\epsilon) + \epsilon^2 l_2(z,\epsilon) + \ldots. \tag{6.12}$$

Equation (6.9) enables $n(z)$ to be expressed in terms of l_0. Comparing (6.11) with (3.14) we can see that but for a difference of sign in one term† we could conclude that $l_0(z)$ was double $f_0(z)$. Like (3.14), (6.11) can be solved in powers of $(\ln\epsilon)^{-1}$, and the dominant term will in fact be $2f_{01}(z)$, since the difference in sign occurs in a term of smaller logarithmic order.

The total force on the body is a drag in the negative x direction given by

$$D = 8\pi\mu\int_\kappa^\lambda l(z)\,dz$$

$$= 8\pi\mu Ua\left\{-(\ln\epsilon)^{-1} - \frac{1}{2}\left(\int_{-a}^a \ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{R(z)}\,dz + 1\right)(\ln\epsilon)^{-2} + O[(\ln\epsilon)^{-3}]\right\}. \tag{6.13}$$

† This is the term whose sign was calculated incorrectly by Taylor (1969), with the consequences described in the introduction.

Comparing this result with (3.24) we see that the ratio of the forces in the transverse and axial cases is

$$\frac{D_{\text{trans}}}{D_{\text{ax}}} = 2+2(\ln\epsilon)^{-1}+O[(\ln\epsilon)^{-2}]. \tag{6.14}$$

This shows the expected factor 2 for the dominant term; we also have the surprising result that the $(\ln\epsilon)^{-1}$ term is also independent of the way in which the cross-sectional radius varies along the length.

In the special case of the spheroid, (6.11) can be solved exactly, as in the axial case, yielding

$$l_0(z) = U/(2\ln(2/\epsilon)+1), \tag{6.15}$$

$$D = 8\pi\mu Ua/(\ln(2/\epsilon)+\tfrac{1}{2})+O(\epsilon^2\ln\epsilon), \tag{6.16}$$

in agreement with Lamb (1932).

To calculate the couple on the body we need to know the local force acting on an element of the surface. In fact, to the order to which we work in this section, this is just the force due to the local stokeslet strength. To see this, we note that, by the same approximation procedure as before, the velocities near the body are

$$q_r = \left[\left\{2\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{r}+2-\frac{\epsilon^2R^2}{r^2}\right\}l_0(z)+\int_\kappa^\lambda\frac{l_0(\hat{z})-l_0(z)}{|\hat{z}-z|}\,d\hat{z}-U\right]\cos\theta$$
$$+O(\epsilon^2 l_0\ln\epsilon,\ r^2 l_0\ln r), \tag{6.17}$$

$$q_\theta = \left[U-\left\{2\ln\frac{2(a^2-z^2)^{\frac{1}{2}}}{r}+\frac{\epsilon^2R^2}{r^2}\right\}l_0(z)-\int_\kappa^\lambda\frac{l_0(\hat{z})-l_0(z)}{|\hat{z}-z|}\,d\hat{z}\right]\sin\theta$$
$$+O(\epsilon^2 l_0\ln\epsilon,\ r^2 l_0\ln r), \tag{6.18}$$

$$q_z = O(\epsilon^2 l_0/r,\ rl_0\ln r). \tag{6.19}$$

Here we have substituted for $m(z)$ from (6.10). From (6.17) and (6.18) we obtain an expression for the pressure:

$$p = 4\mu l_0(z)\cos\theta/r+O(\epsilon^2 l_0\ln\epsilon/r,\ rl_0\ln r). \tag{6.20}$$

The viscous stress on the surface is in the θ direction to lowest order, and is given by

$$\tau = \mu\left(\frac{\partial q_\theta}{\partial r}+\frac{1}{r}\frac{\partial q_r}{\partial\theta}-\frac{1}{r}q_\theta\right)$$
$$= 4\mu l_0(z)\sin\theta/r+O(\epsilon^2 l_0\ln\epsilon/r,\ rl_0\ln r). \tag{6.21}$$

Hence the total stress to lowest order is $4\mu l_0(z)/\epsilon R$ acting in the negative x direction. This gives rise to the total force (6.13) and to a couple

$$G_y = 8\pi\mu\int_\kappa^\lambda zl_0(z)\,dz, \tag{6.22}$$

which is $O[(\ln\epsilon)^{-2}]$. The Stokes doublets do not contribute to the couple to this order. If the body is symmetric about $z = 0$ the couple (6.22) vanishes, a result expected from the reversibility of Stokes flow.

Equations (6.4) to (6.6) can be expanded to second order along the lines of § 4, and there seems to be no reason in principle why the expansion could not be carried out to any order. The analysis of § 5 can also be carried out in the transverse-flow case; the second-order corrections to the endpoints of the stokeslet and source doublet distributions are the same as those found for axial flow.† The details will not be given here.

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Appendix A

In this appendix we give the details of the method for the expansion of the integrals occurring in (2.6) and (2.7); we treat the first integral in (2.6). The method follows closely the procedure of Handelsman & Keller (1967).

From the expressions

$$E_n k = k_0 + k_1/\sigma + \ldots + k_n/\sigma^n, \tag{A 1}$$

$$H_n h = h_0 + h_1 s + \ldots + h_n s^n, \tag{A 2}$$

where the h_m and k_m are obtained from the Taylor expansion

$$h(s) = \sum_{r=0}^{\infty} \frac{(-1)^r f^{(r)}(z)}{r!} s^r \tag{A 3}$$

and the binomial expansion

$$k(\sigma) = \left(2 + \frac{R^2}{\sigma^2}\right) . \left(1 + \frac{R^2}{\sigma^2}\right)^{-\frac{3}{2}}$$

$$= \sum_{j=0}^{\infty} a_j (R^2/\sigma^2)^j, \quad \text{say}, \tag{A 4}$$

we obtain, by induction,

$$H_n h . E_n k = \sum_{r=0}^{n} \left\{\frac{k_r}{\sigma^r} H_r h + h_r s^r E_{r-1} k\right\}. \tag{A 5}$$

Hence

$$C_{nn}\Phi = \sum_{r=0}^{n} \frac{1}{s}\left\{\frac{k_r}{\sigma^r}(h - H_r h) + h_r s^r (k - E_{r-1} k)\right\}, \tag{A 6}$$

and on introducing (A 3) and (A 4) we find

$$\begin{aligned} C_{2N,\,2N}\Phi = {} & \sum_{n=0}^{N} \left\{f(z-s) - \sum_{r=0}^{2n} \frac{f^{(r)}(z)\,(-s)^r}{r!}\right\} \frac{1}{s} a_n \left(\frac{\epsilon^2 R^2}{s^2}\right)^n \\ & + \sum_{n=0}^{N} \left\{\frac{2s^2 + \epsilon^2 R^2}{(s^2+\epsilon^2 R^2)^{\frac{3}{2}}} - \sum_{j=0}^{n-1} a_j \frac{1}{s}\left(\frac{\epsilon^2 R^2}{s^2}\right)^j\right\} \frac{f^{(2n)}(z)\, s^{2n}}{(2n)!} \\ & + \sum_{n=0}^{N-1} \left\{\frac{2s^2 + \epsilon^2 R^2}{(s^2+\epsilon^2 R^2)^{\frac{3}{2}}} - \sum_{j=0}^{n} a_j \frac{1}{s}\left(\frac{\epsilon^2 R^2}{s^2}\right)^j\right\} \frac{f^{(2n+1)}(z)\,(-s)^{2n+1}}{(2n+1)!}. \end{aligned} \tag{A 7}$$

† Higher-order knowledge of $m(z)$ is needed to establish this result. To find the endpoints of the Stokes doublet distribution it would be necessary to solve for $l_2(z)$.

With $s = \epsilon\sigma$, we define

$$\Psi_n(\sigma, z) = \frac{2\sigma^2 + R^2}{(\sigma^2 + R^2)^{\frac{3}{2}}} - \frac{1}{\sigma}\sum_{j=0}^{n} a_j \left(\frac{k^2}{\sigma^2}\right)^j, \tag{A 8}$$

and obtain for the first integral in (3.3)

$$\int_0^{z-\alpha} \frac{2s^2 + \epsilon^2 R^2}{(s^2 + \epsilon^2 R^2)^{\frac{3}{2}}} f(z-s)\, ds$$

$$= \sum_{n=0}^{N} a_n(\epsilon^2 R^2)^n \mathscr{G}_n f + f(z)\int_0^{z-\alpha} \frac{2s^2+\epsilon^2 R^2}{(s^2+\epsilon^2 R^2)^{\frac{3}{2}}}\, ds - \sum_{n=0}^{N-1} \frac{\epsilon^{2n+1}}{(2n+1)!} f^{(2n+1)}(z)\, P_{2n+1}(z,\epsilon)$$
$$+ \sum_{n=0}^{N-1} \frac{\epsilon^{2n+2}}{(2n+2)!} f^{(2n+2)}(z)\, Q_{2n+2}(z,\epsilon) + S_{2N}, \tag{A 9}$$

where
$$\mathscr{G}_n f = \int_0^{z-\alpha} s^{-2n-1}\left\{f(z-s) - \sum_{r=0}^{2n} \frac{f^{(r)}(z)\,(-s)^r}{r!}\right\} ds,$$
$$P_{2n+1} = \int_0^{(z-\alpha)/\epsilon} \sigma^{2n+1}\Psi_n(\sigma, z)\, d\sigma,$$
$$Q_{2n+2} = \int_0^{(z-\alpha)/\epsilon} \sigma^{2n+2}\Psi_n(\sigma, z)\, d\sigma,$$

and where the remainder S_{2N} is bounded by means of (3.10):†

$$|S_{2N}| = \left|\int_0^{z-\alpha} (\Phi - C_{2N,\,2N}\,\Phi)\, ds\right|$$
$$\leqslant |f^{(2N+1)}|_{\max} \int_0^{(z-\alpha)/\epsilon} \frac{\epsilon^{2N+1}}{\sigma+1}\, d\sigma$$
$$= A\,|f^{(2N+1)}|_{\max}\, \epsilon^{2N+1} \ln(1 + (z-\alpha)/\epsilon). \tag{A 10}$$

Note that the integral defining P_{2n+1} converges since $\sigma^{2n+1}\Psi_n$ is $O(1)$ for $\sigma \to 0$ and $O(\sigma^{-2})$ for $\sigma \to \infty$; that for Q_{2n+2} is $O[\ln(1 + (z-\alpha)/\epsilon)]$. The same analysis may be applied to the second integral in (3.3); corresponding to (A 9) we have

$$\int_0^{\beta-z} \frac{2s^2 + \epsilon^2 R^2}{(s^2 + \epsilon^2 R^2)^{\frac{3}{2}}} f(z+s)\, ds$$

$$= \sum_{n=0}^{N} a_r(\epsilon^2 R^2)^n \tilde{\mathscr{G}}_n f + f(z)\int_0^{\beta-z} \frac{2s^2+\epsilon^2 R^2}{(s^2+\epsilon^2 R^2)^{\frac{3}{2}}}\, ds + \sum_{n=0}^{N-1} \frac{\epsilon^{2n+1}}{(2n+1)!} f^{(2n+1)}(z)\, \tilde{P}_{2n+1}(z,\epsilon)$$
$$+ \sum_{n=0}^{N-1} \frac{\epsilon^{2n+2}}{(2n+2)!} f^{(2n+1)}(z)\, \tilde{Q}_{2n+2}(z,\epsilon) + \tilde{S}_{2N}, \tag{A 11}$$

where
$$\tilde{\mathscr{G}}_n f = \int_0^{\beta-z} s^{-2n-1}\left\{f(z+s) - \sum_{r=0}^{2n} \frac{f^{(r)}(z)\, s^r}{r!}\right\} ds,$$
$$\tilde{P}_{2n+1} = \int_0^{(\beta-z)/\epsilon} \sigma^{2n+1}\Psi_n(\sigma, z)\, d\sigma,$$
$$\tilde{Q}_{2n+2} = \int_0^{(\beta-z)/\epsilon} \sigma^{2n+2}\Psi_n(\sigma, z)\, d\sigma,$$

and
$$|\tilde{S}_{2N}| \leqslant A\,|f^{(2N+1)}|_{\max}\, \epsilon^{2N+1} \ln(1 + (\beta-z)/\epsilon).$$

† Handelsman & Keller (1967) obtain a sharper estimate for S_{2N}, but this is no better than our result (A 15) for the total error.

The total error $S_{2N}+\tilde{S}_{2N}$ is in fact smaller than the foregoing results suggest. We have

$$S_{2N} = \int_0^{(z-\alpha)/\epsilon} \left\{ f(z-s) - \sum_{r=0}^{2N} \frac{f^{(r)}(z)\,(-s)^r}{r!} \right\} \Psi_N(\sigma, z)\, d\sigma, \tag{A 12}$$

$$\tilde{S}_{2N} = \int_0^{(\beta-z)/\epsilon} \left\{ f(z+s) - \sum_{r=0}^{2N} \frac{f^{(r)}(z)\, s^r}{r!} \right\} \Psi_N(\sigma, z)\, d\sigma, \tag{A 13}$$

so that

$$S_{2N}+\tilde{S}_{2N} = \frac{\epsilon^{2N+1}}{(2N+1)!} f^{(2N+1)}(z) \int_{(z-\alpha)/\epsilon}^{(\beta-z)/\epsilon} \sigma^{2N+1}\Psi_N(\sigma, z)\, d\sigma + T_{2N}, \tag{A 14}$$

where

$$|T_{2N}| \leqslant \frac{\epsilon^{2N+2}}{(2N+2)!} |f^{(2N+2)}|_{\max} .\, 2 \int_0^{\max((\beta-z)/\epsilon,\,(z-\alpha)/\epsilon)} \sigma^{2N+2} |\Psi_N(\sigma, z)|\, d\sigma.$$

Now in (A 14), $\sigma^{2N+1}\Psi_N$ is $O(\sigma^{-2})$ for $\sigma \to \infty$; hence for z bounded away from α and β

$$S_{2N}+\tilde{S}_{2N} = O(\epsilon^{2N+2} f^{(2N+1)}) + O(\epsilon^{2N+2} f^{(2N+2)} \ln \epsilon). \tag{A 15}$$

In the text it is assumed for brevity that f and its derivatives are all of the same order, so that

$$S_{2N}+\tilde{S}_{2N} = O(\epsilon^{2N+2} f \ln \epsilon). \tag{A 16}$$

Similar analysis may be applied to the other three integrals in (2.6) and (2.7) and asymptotic expansions to arbitrary order obtained for them; it has not been thought worthwhile to record the details here. The lowest-order forms are given in appendix B.

Appendix B

The work of §§ 4 and 6 requires the expansions of integrals of the form

$$\int_{b_1}^{b_2} \frac{\phi(\hat{z})\, d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{1}{2}n}}, \quad \int_{b_1}^{b_2} \frac{(z-\hat{z})\,\phi(\hat{z})\, d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{1}{2}n}}$$

for $n = 1, 3, 5$. This can be accomplished to any order by the procedure described in § 3 and appendix A, or slight modifications of it. We require only the lowest-order expansions.

The integrals are decomposed as

$$\int_{b_1}^{b_2} \frac{\phi(\hat{z})\, d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{1}{2}n}} = \int_0^{b_2-z} \frac{\phi(z+s)\, ds}{(s^2+\epsilon^2R^2)^{\frac{1}{2}n}} + \int_0^{z-b_1} \frac{\phi(z-s)\, ds}{(s^2+\epsilon^2R^2)^{\frac{1}{2}n}}, \tag{B 1}$$

$$\int_{b_1}^{b_2} \frac{(z-\hat{z})\,\phi(\hat{z})\, d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{1}{2}n}} = -\int_0^{b_2-z} \frac{s\phi(z+s)\, ds}{(s^2+\epsilon R^2)^{\frac{1}{2}n}} + \int_0^{z-b_1} \frac{s\phi(z-s)\, ds}{(s^2+\epsilon^2R^2)^{\frac{1}{2}n}}. \tag{B 2}$$

Where the degree of the denominator is one greater than the numerator, as in (B 1) with $n = 1$, the procedure is just as in § 3; this gives for the first integral on the right

$$\int_0^{b_2-z} \frac{\phi(z+s)\, ds}{(s^2+\epsilon^2R^2)^{\frac{1}{2}}} = \phi(z) \int_0^{b_2-z} \frac{ds}{(s^2+\epsilon^2R^2)^{\frac{1}{2}}} + \int_0^{b_2-z} \frac{\phi(z+s)-\phi(z)}{s}\, ds + O(\epsilon\phi \ln \epsilon), \tag{B 3}$$

whence, with the error improved as in appendix A, we obtain (4.7).

Where the degree of the denominator is two greater than the numerator, as in (B 2) with $n = 3$, we have to carry the inner expansion of the integrand one stage further than the outer, giving

$$\int_0^{b_2-z} \frac{s\phi(z+s)\,ds}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}} = \phi(z)\int_0^{b_2-z}\frac{s\,ds}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}} + \phi'(z)\int_0^{b_2-z}\frac{s^2\,ds}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}}$$
$$+\int_0^{b_2-z}\frac{(z+s)-\phi(z)-s\phi'(z)}{s^2}\,ds$$
$$+\int_0^{b_2-z}\{\phi(z+s)-\phi(z)-s\phi'(z)\}\left\{\frac{s}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}}-\frac{1}{s^2}\right\}ds$$
$$=\phi(z)\left(\frac{1}{\epsilon R}-\frac{1}{a-z}\right)+\phi'(z)\left\{\ln\frac{2(a-z)}{\epsilon R}-1\right\}$$
$$+\int_0^{b_2-z}\frac{\phi(z+s)-\phi(z)-s\phi'(z)}{s^2}\,ds+O(\epsilon\phi\ln\epsilon), \tag{B 4}$$

which gives (4.6).

An extension of this idea gives

$$\int_0^{b_2-z}\frac{\phi(z+s)\,ds}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}} = \phi(z)\int_0^{b_2-z}\frac{ds}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}}$$
$$+\phi'(z)\int_0^{b_2-z}\frac{s\,ds}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}}+\phi''(z)\int_0^{b_2-z}\frac{s^2\,ds}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}}$$
$$+\int_0^{b_2-z}\frac{\phi(z+s)-\phi(z)-s\phi'(z)-\frac{1}{2}s^2\phi''(z)}{s^3}\,ds$$
$$+\int_0^{b_2-z}\{\phi(z+s)-\ldots-\tfrac{1}{2}s^2\phi''(z)\}\left\{\frac{1}{(s^2+\epsilon^2R^2)^{\frac{3}{2}}}-\frac{1}{s^3}\right\}ds, \tag{B 5}$$

whence

$$\int_{b_1}^{b_2}\frac{\phi(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{3}{2}}}$$
$$=\phi(z)\left\{\frac{2}{\epsilon^2R^2}-\frac{1}{2}\frac{a^2+z^2}{(a^2-z^2)^2}\right\}-\phi'(z)\frac{2z}{a^2-z^2}+\tfrac{1}{2}\phi''(z).2\ln\frac{(a^2-z^2)^{\frac{1}{2}}}{\epsilon R}$$
$$+\int_{b_1}^{b_2}\frac{\phi(\hat{z})-\phi(z)-(\hat{z}-z)\,\phi'(z)-\frac{1}{2}(\hat{z}-z)^2\,\phi''(z)}{|\hat{z}-z|^3}\,d\hat{z}+O(\epsilon^2\phi\ln\epsilon). \tag{B 6}$$

Where the numerator and denominator have the same degree, no inner expansion at all is required to the lowest order; we have simply

$$\int_0^{b\ -z}\frac{s\phi(z+s)\,ds}{(s^2+\epsilon^2R^2)^{\frac{1}{2}}}=\int_0^{b_2-z}\phi(z+s)\,ds+O(\epsilon\phi\ln\epsilon), \tag{B 7}$$

whence (4.8) follows.

The expansion for the integrals with $n = 5$ are lengthier, and will not be given in full here; the work of § 6 requires only the leading terms:

$$\int_{b_1}^{b_2}\frac{\phi(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{5}{2}}}=\frac{4}{3}\frac{\phi(z)}{\epsilon^4R^4}+O(\epsilon^{-2}\phi\ln\epsilon), \tag{B 8}$$

$$\int_{b_1}^{b_2}\frac{(z-\hat{z})\,\phi(\hat{z})\,d\hat{z}}{\{(z-\hat{z})^2+\epsilon^2R^2\}^{\frac{5}{2}}}=\frac{2}{3}\frac{\phi'(z)}{\epsilon^2R^2}+O(\phi\ln\epsilon). \tag{B 9}$$

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